FORMULAIRE DE MÉCANIQUE DES FLUIDES

Opérateurs classiques

en coordonnées cartésiennes

$$\overrightarrow{\text{grad}}\ f(x,y,z) = \vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\text{div}\ \vec{A} = \vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{\text{rot}}\ \vec{A} = \vec{\nabla}\wedge\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{e}_z$$

$$\Delta f = \vec{\nabla}\cdot\left(\vec{\nabla} f\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta\vec{A} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \vec{\nabla}\wedge\left(\vec{\nabla}\wedge\vec{A}\right) = \Delta A_x\ \vec{e}_x + \Delta A_y\ \vec{e}_y + \Delta A_z\ \vec{e}_z$$

en coordonnées cylindriques

$$\vec{\nabla} f(r,\theta,z) = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\wedge\vec{A} = \frac{1}{r}\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial}{\partial z}\left(rA_\theta\right)\right)\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial \theta}\right)\vec{e}_z$$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta\vec{A} = \left(\Delta A_r - \frac{1}{r^2}A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\right)\vec{e}_r + \left(\Delta A_\theta - \frac{1}{r^2}A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta}\right)\vec{e}_\theta + \Delta A_z\ \vec{e}_z$$

en coordonnées sphériques

$$\vec{\nabla} f(r,\theta,\varphi) = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta\sin\theta\right) + \frac{1}{\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)$$

$$\vec{\nabla}\wedge\vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\left(A_\varphi\sin\theta\right) - \frac{\partial A_\theta}{\partial \varphi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial \theta}\right)\vec{e}_\varphi$$

$$\Delta f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}\right)$$

$$\begin{aligned}\Delta\vec{A} = &\left(\Delta A_r - \frac{2}{r^2}A_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta\sin\theta\right) - \frac{2}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\vec{e}_r \\ + &\left(\Delta A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{1}{r^2\sin^2\theta}A_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\vec{e}_\theta \\ + &\left(\Delta A_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial \varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial \varphi} - \frac{1}{r^2\sin^2\theta}A_\varphi\right)\vec{e}_\varphi\end{aligned}$$

propriétés générales

$$\vec{\nabla} \cdot (f\ \vec{A}) = \vec{A} \cdot \vec{\nabla} f + f\ \vec{\nabla} \cdot \vec{A}$$
$$\vec{\nabla} \wedge (f\ \vec{A}) = f\ \vec{\nabla} \wedge \vec{A} + \vec{\nabla} f \wedge \vec{A}$$
$$\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A}) = 0$$
$$\vec{\nabla} \wedge (\vec{\nabla} f) = \vec{0}$$
$$\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Transformation d'intégrales multiples

Formule d'Ostrogradski

$$\iiint_{(\tau)} \vec{\nabla} \cdot \vec{A} \cdot \mathrm{d}\tau = \iint_{(\sigma)} \vec{A} \cdot \vec{n}\ \mathrm{d}\sigma$$

(σ) est une surface fermée, frontière d'un domaine (σ) ;
$\vec{n}$ est le vecteur unitaire normal à (σ) dirigé vers l'extérieur.

Conséquences :

Formule du gradient $\iiint_{(\tau)} \vec{\nabla} f\ \mathrm{d}\tau = \iint_{(\sigma)} f\ \vec{n}\ \mathrm{d}\sigma$

Formule du rotationnel $\iiint_{(\tau)} \vec{\nabla} \wedge \vec{A}\ \mathrm{d}\tau = \iint_{(\sigma)} \vec{n} \wedge \vec{A}\ \mathrm{d}\sigma$

$$\iint_{(\sigma)} \vec{n} \wedge \vec{\nabla} f\ \mathrm{d}\sigma = \int_{(L)} f\ \mathrm{d}\vec{r}$$

Formule de Stokes-Ampère

$$\iint_{(\sigma)} (\vec{\nabla} \wedge \vec{A}) \cdot \vec{n}\ \mathrm{d}\sigma = \int_{(L)} \vec{A} \cdot \mathrm{d}\vec{r}$$

La circulation d'un vecteur $\vec{A}$ le long d'un contour fermé (L) est égale au flux de $\vec{\nabla} \wedge \vec{A}$ à travers une surface(σ) admettant (L) comme frontière.

Fonction de courant et potentiel des vitesses

$$v_x = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v_r = \frac{\partial \varphi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad v_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Potentiel complexe des vitesses :

$$f(z) = \varphi + i\psi \quad \text{avec} \quad z = x + iy \quad \text{ou} \quad z = r\ e^{i\theta}$$

Tenseur des contraintes pour un fluide newtonien incompressible

en coordonnées cartésiennes

$$\sigma_{xx} = -p + 2\mu \frac{\partial v_x}{\partial x} \qquad \tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$
$$\sigma_{yy} = -p + 2\mu \frac{\partial v_y}{\partial y} \qquad \tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)$$
$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z} \qquad \tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

en coordonnées cylindriques

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r} \qquad \tau_{r\theta} = \tau_{\theta r} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)$$
$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \qquad \tau_{\theta z} = \tau_{z\theta} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)$$
$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z} \qquad \tau_{zr} = \tau_{rz} = \mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)$$

Equation de *Navier-Stokes* pour un fluide newtonien incompressible

en coordonnées cartésiennes

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right)$$
$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right)$$
$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) - \rho g$$

en coordonnées cylindriques

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right)$$
$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right)$$
$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) - \rho g$$